Pergamon

Computers Math. Applic. Vol. 31, No. 6, pp. 117-120, 1996
Copyright©1996 Elsevier Science Ltd
Printed in Great Britain. All rights reserved
0898-1221/96 \$15.00 + 0.00

S0898-1221(96)00010-7

Solution of Coupled Nonlinear Partial Differential Equations by Decomposition

G. ADOMIAN
General Analytics Corporation, 155 Clyde Rd.
Athens, GA 30605, U.S.A.

(Received April 1995; accepted June 1995)

Abstract—Solution of a coupled system of nonlinear partial differential equations is demonstrated for uncoupled boundary conditions using the decomposition method. Linear systems, single partial differential equations, ordinary differential equations or systems become special cases.

Keywords—Modified decomposition, Coupled nonlinear partial differential equations, Double decomposition, Adomian polynomials, Convergence acceleration.

Consider the system

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} + \alpha uv &= g_1(x,t),\\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial t^2} + \beta uv &= g_2(x,t).\end{aligned}$$

Given (uncoupled) boundary conditions are

$$\begin{aligned} u(x_1,t) &= \xi_1(t), & u(x,0) &= \tau_1(x),\\ u(x_2,t) &= \xi_2(t), & u_t(x,0) &= \tau_2(x),\\ v(x_1,t) &= \eta_1(t), & v(x,0) &= \sigma_1(x),\\ v(x_2,t) &= \eta_2(t), & v_t(x,0) &= \sigma_2(x).\end{aligned}$$

Since we will solve for $\frac{\partial^2 u}{\partial x^2}$ using the spatial format, the initial conditions are not used. Solving for $\frac{\partial^2 u}{\partial t^2}$ with the initial conditions will yield the same solution [1].

In operator form using the given conditions on x,

$$\begin{aligned} Lu &= g_1(x,t) - \left(\frac{\partial^2}{\partial t^2}\right) u - \alpha uv,\\ Lv &= g_2(x,t) - \left(\frac{\partial^2}{\partial t^2}\right) v - \beta uv,\end{aligned}$$

where $L = \frac{\partial^2}{\partial x^2}$ and L^{-1} is a two-fold indefinite integration. Operating with L^{-1}

$$\begin{aligned} u &= u_0 - L^{-1}\left(\frac{\partial^2}{\partial t^2}\right) u - L^{-1}\alpha uv,\\ v &= v_0 - L^{-1}\left(\frac{\partial^2}{\partial t^2}\right) v - L^{-1}\beta uv,\end{aligned}$$

This work was supported by the Army Research Laboratory, White Sands Missile Range, NM under a contract administered by the Battelle Institute.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

and

$$u_0 = A_0(t) + xB_0(t) + L^{-1}g_1(x,t),$$
$$v_0 = C_0(t) + xD_0(t) + L^{-1}g_2(x,t).$$

We write $u = \sum_{n=0}^{\infty} u_n$ and $v = \sum_{n=0}^{\infty} v_n$. The uv term can be written

$$uv = \sum_{n=0}^{\infty} \left[\sum_{\lambda=0}^{n} u_{n-\lambda} v_\lambda \right],$$

where the bracketed quantity is the Adomian polynomial of order n (see footnote[1]). The initial terms u_0, v_0 have been given, and terms after $n = 0$ are given by

$$u_n = A_n(t) + B_n(t)x - L^{-1}\left(\frac{\partial^2}{\partial t^2}\right) u_{n-1} - L^{-1}\alpha \sum_{\lambda=0}^{n-1} u_{n-1-\lambda} v_\lambda,$$
$$v_n = C_n(t) + D_n(t)x - L^{-1}\left(\frac{\partial^2}{\partial t^2}\right) v_{n-1} - L^{-1}\beta \sum_{\lambda=0}^{n-1} u_{n-1-\lambda} v_\lambda.$$

The approximant $\varphi_1[u] = u_0$ and $\varphi_1[v] = v_0$. $\varphi_{\lambda+1}[u] = \sum_{n=0}^{\lambda} u_n$ and $\varphi_{\lambda+1}[v] = \sum_{n=0}^{\lambda} v_n$ converge to $u = \sum_{n=0}^{\infty} u_n$ and $v = \sum_{n=0}^{\infty} v_n$.

The successive approximants must satisfy the relevant boundary conditions, and hence for φ_1 or u_0,

$$u_0(x_1,t) = \varphi_1[u(x_1,t)] = \xi_1(t),$$
$$u_0(x_2,t) = \varphi_1[u(x_2,t)] = \xi_2(t),$$
$$v_0(x_1,t) = \varphi_1[v(x_1,t)] = \eta_1(t),$$
$$v_0(x_2,t) = \varphi_1[v(x_2,t)] = \eta_2(t).$$

Since each successive approximant $\varphi_1, \varphi_2, \dots$ must satisfy the given conditions, we have for all $n > 0$:

$$\varphi_n[u(x_1,t)] = \xi_1(t),$$
$$\varphi_n[u(x_2,t)] = \xi_2(t),$$
$$\varphi_n[v(x_1,t)] = \eta_1(t),$$
$$\varphi_n[v(x_2,t)] = \eta_2(t).$$

We observe that $\varphi_{n+1}[u] = \varphi_n[u] + u_n$ and $\varphi_{n+1}[v] = \varphi_n[v] + v_n$, and consequently for $n \geq 1$,

$$u_n(x_1,t) = u_n(x_2,t) = 0,$$
$$v_n(x_1,t) = v_n(x_2,t) = 0.$$

Consequently, for $\varphi_1 = u_0$,

$$A_0(t) + x_1B_0(t) + L^{-1}g_1(x_1,t) = \xi_1(t),$$
$$A_0(t) + x_2B_0(t) + L^{-1}g_1(x_2,t) = \xi_2(t),$$
$$C_0(t) + x_1D_0(t) + L^{-1}g_2(x_1,t) = \eta_1(t),$$
$$C_0(t) + x_2D_0(t) + L^{-1}g_2(x_2,t) = \eta_2(t).$$

[1]The algorithm for a function of two variables is unnecessary because of the simple product function. Thus uv is replaced by $\sum_{n=0}^{\infty} A_n = \sum_{n=0}^{\infty} \sum_{\lambda=0}^{n} u_{n-\lambda} v_\lambda$.

Let us rewrite this as

$$\begin{aligned}
A_0(t) + x_1 B_0(t) &= \xi_1^{(0)}(t) \equiv \xi_1(t) - L^{-1} g_1(x_1, t), \\
A_0(t) + x_2 B_0(t) &= \xi_2^{(0)}(t) \equiv \xi_2(t) - L^{-1} g_1(x_2, t), \\
C_0(t) + x_1 D_0(t) &= \eta_1^{(0)}(t) \equiv \eta_1(t) - L^{-1} g_2(x_1, t), \\
C_0(t) + x_2 D_0(t) &= \eta_2^{(0)}(t) \equiv \eta_2(t) - L^{-1} g_2(x_2, t).
\end{aligned} \tag{1}$$

We can also write as a result of the decomposition

$$\begin{aligned}
\xi_1^{(n)}(t) &= L^{-1}\left(\frac{\partial^2}{\partial t^2}\right) u_{n-1}(x_1, t) + L^{-1}\alpha \sum_{\lambda=0}^{n} u_{n-\lambda}(x_1, t) v_\lambda(x_1, t), \\
\xi_2^{(n)}(t) &= L^{-1}\left(\frac{\partial^2}{\partial t^2}\right) u_{n-1}(x_2, t) + L^{-1}\alpha \sum_{\lambda=0}^{n} u_{n-\lambda}(x_2, t) v_\lambda(x_2, t), \\
\eta_1^{(n)}(t) &= L^{-1}\left(\frac{\partial^2}{\partial t^2}\right) v_{n-1}(x_1, t) + L^{-1}\beta \sum_{\lambda=0}^{n} u_{n-\lambda}(x_1, t) v_\lambda(x_1, t), \\
\eta_2^{(n)}(t) &= L^{-1}\left(\frac{\partial^2}{\partial t^2}\right) v_{n-1}(x_2, t) + L^{-1}\beta \sum_{\lambda=0}^{n} u_{n-\lambda}(x_2, t) v_\lambda(x_2, t).
\end{aligned}$$

Also corresponding to (1), we write for determination of the constants of integration for $n \geq 1$

$$\begin{aligned}
A_n(t) + x_1 B_n(t) &= \xi_1^{(n)}(t), \\
A_n(t) + x_2 B_n(t) &= \xi_2^{(n)}(t), \\
C_n(t) + x_1 D_n(t) &= \eta_1^{(n)}(t), \\
C_n(t) + x_2 D_n(t) &= \eta_2^{(n)}(t).
\end{aligned}$$

We can now write

$$\begin{aligned}
\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} \cdot \begin{bmatrix} A_0(t) \\ B_0(t) \end{bmatrix} &= \begin{bmatrix} \xi_1^{(0)}(t) \\ \xi_2^{(0)}(t) \end{bmatrix}, \\
\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} \cdot \begin{bmatrix} C_0(t) \\ D_0(t) \end{bmatrix} &= \begin{bmatrix} \eta_1^{(0)}(t) \\ \eta_2^{(0)}(t) \end{bmatrix}, \\
\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} \cdot \begin{bmatrix} A_n(t) \\ B_n(t) \end{bmatrix} &= \begin{bmatrix} \xi_1^{(n)}(t) \\ \xi_2^{(n)}(t) \end{bmatrix}, \\
\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} \cdot \begin{bmatrix} C_n(t) \\ D_n(t) \end{bmatrix} &= \begin{bmatrix} \eta_1^{(n)}(t) \\ \eta_2^{(n)}(t) \end{bmatrix}.
\end{aligned} \tag{2}$$

Inversion of the first matrix to $(1/(x_2 - x_1))\begin{bmatrix} x_2 & -x_1 \\ -1 & 1 \end{bmatrix}$ allows determination of the integration constants, e.g.,

$$\begin{aligned}
A_0(t) &= \frac{x_2 \xi_1^{(0)}(t) - x_1 \xi_2^{(0)}(t)}{x_2 - x_1}, \\
B_0(t) &= \frac{\xi_2^{(0)}(t) - \xi_1^{(0)}(t)}{x_2 - x_1}, \\
C_0(t) &= \frac{x_2 \eta_1^{(0)}(t) - x_1 \eta_2^{(0)}(t)}{x_2 - x_1}, \\
D_0(t) &= \frac{\eta_2^{(0)}(t) - \eta_1^{(0)}(t)}{x_2 - x_1},
\end{aligned}$$

so that u_0 and v_0 are determined.

To calculate the u_1 and v_1, or the $\varphi_2[u]$ and $\varphi_2[v]$ approximants, we continue with the matrix equations in (2)

$$\begin{aligned} A_n(t) &= \frac{x_2\xi_1^{(n)}(t) - x_1\xi_2^{(n)}(t)}{x_2 - x_1}, \\ B_n(t) &= \frac{\xi_2^{(n)}(t) - \xi_1^{(n)}(t)}{x_2 - x_1}, \\ C_n(t) &= \frac{x_2\eta_1^{(n)}(t) - x_1\eta_2^{(n)}(t)}{x_2 - x_1}, \\ D_n(t) &= \frac{\eta_2^{(n)}(t) - \eta_1^{(n)}(t)}{x_2 - x_1}. \end{aligned}$$

We have

$$\begin{aligned} u_1 &= A_1(t) + xB_1(t) - L^{-1}\left(\frac{\partial^2}{\partial t^2}\right)u_0 - L^{-1}\alpha u_0 v_0, \\ v_1 &= C_1(t) + xD_1(t) - L^{-1}\left(\frac{\partial^2}{\partial t^2}\right)v_0 - L^{-1}\beta u_0 v_0, \end{aligned}$$

and

$$\begin{aligned} \varphi_2[u] &= \varphi_1[u] + u_1, \\ \varphi_2[v] &= \varphi_1[v] + v_1. \end{aligned}$$

We can continue to obtain components u_n and v_n and approximants and

$$\begin{aligned} u_n &= A_n(t) + xB_n(t) - L^{-1}\left(\frac{\partial^2}{\partial t^2}\right)u_{n-1} - L^{-1}\alpha\sum_{\lambda=0}^{n-1} u_{n-1-\lambda}v_\lambda, \\ v_n &= C_n(t) + xD_n(t) - L^{-1}\left(\frac{\partial^2}{\partial t^2}\right)v_{n-1} - L^{-1}\beta\sum_{\lambda=0}^{n-1} u_{n-1-\lambda}v_\lambda, \end{aligned}$$

so that the approximants $\varphi_{n+1}[u] = \varphi_n[u] + u_n$ and $\varphi_{n+1}[v] = \varphi_n[v] + v_n$ can be formed. These approximants for increasing n provide an increasingly accurate solution. Different nonlinearities than the uv term simply require the Adomian polynomials generated for the specific term. For example, if we denote the polynomial by $\hat{A}_n$ to distinguish it from the integration constant A_n, we would write u^2 as $\sum_{n=0}^{\infty}\hat{A}_n\{u^2\}$ where $\hat{A}_n = \sum_{\lambda=0}^{n} u_{n-\lambda}u_\lambda$. Similarly, u^3 will be written $\sum_{n=0}^{\infty}\hat{A}_n\{u^3\}$ where $\hat{A}_n = \sum_{\lambda=0}^{n}\sum_{\mu=0}^{\lambda} u_{n-\lambda}u_{\lambda-\mu}u_\mu$ and we can write the polynomials for products such as u^3v^2.

More general functions are treated by algorithms given in [1]. The technique of double decomposition [1] can be added to the analysis so that the boundary-value problem can be reformulated as an initial-value problem [1] after calculating a certain number of terms to accelerate convergence minimizing further computation for matching boundaries.

REFERENCES

1. G. Adomian, *Solving Frontier Problems of Physics: The Decomposition Method*, Kluwer Academic, (1994).